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The canonical connection of a bi-Lagrangian manifold

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Abstract

We prove that the canonical connection of a bi-Lagrangian manifold introduced by Hess is a Levi-Civita connection, showing that a bi-Lagrangian manifold (i.e. a symplectic manifold endowed with two transversal Lagrangian foliations) is endowed with a canonical semi-Riemannian metric.

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1. Introduction

Lagrangian foliations on symplectic manifolds are used in geometric quantization. As is well known, the existence of a connection canonically attached to a symplectic manifold is an important tool in obtaining a deformation quantization [1–3]. A bi-Lagrangian manifold (i.e. a symplectic manifold endowed with two transversal Lagrangian foliations) admits a canonical symplectic connection, which has been introduced by Hess in [1], and was used by several authors [4, 5].

Here, we shall study geometric properties of a symplectic manifold endowed with a Lagrangian distribution, and with two transversal Lagrangian distributions. We shall prove the following important results.

Theorem 1. *A symplectic manifold endowed with a Lagrangian distribution admits infinitely many Lagrangian distributions.*

Theorem 2. *If a Kähler manifold admits a Lagrangian foliation $\mathcal{F}$ which is preserved by the Levi-Civita connection ∇, then the orthogonal distribution to $\mathcal{F}$ defines a foliation $\mathcal{F}^{\perp}$ and the canonical connection of the bi-Lagrangian structure defined by $\mathcal{F}$ and $\mathcal{F}^{\perp}$ is ∇.*

Theorem 3. *A bi-Lagrangian manifold is endowed with a canonical semi-Riemannian metric, whose Levi-Civita connection coincides with the canonical connection of the bi-Lagrangian manifold.*

Theorem 2 proves that in some situations the canonical connection is the Levi-Civita connection of a Riemannian metric. Theorem 3 states that the canonical connection is always the Levi-Civita connection of a canonical semi-Riemannian metric, thus allowing us the best comprehension of this connection.

In fact, we shall prove that the bi-Lagrangian manifolds are exactly the para-Kähler manifolds, as we shall show in the appendix. Thus, one can build a bridge between these theories.

All the manifolds throughout the paper will be assumed to be smooth. The Lie algebra of vector fields of a manifold M will be denoted as $\mathcal{X}(M)$. A Riemannian metric will be denoted as G, whereas a semi-Riemannian metric of signature (n, n) will be denoted as g. On the other hand, automorphisms of $\mathcal{X}(M)$ of square $-\mathrm{id}$ (respectively id) will be denoted as J (respectively F).

2. Preliminaries

Let (M, ω) be a symplectic manifold, with $\dim M = 2n$. In the last few years, the following definitions have been introduced:

(a) A *Lagrangian distribution* on M is an n-dimensional distribution $\mathcal{D}$ such that $\omega(X, Y) = 0$ for all vector fields $X, Y \in \mathcal{D}$. Such a Lagrangian distribution is also called an *almost cotangent structure* [6].

(b) A foliation $\mathcal{F}$ on M is said to be a *Lagrangian foliation* if its leaves are Lagrangian submanifolds, i.e. each leaf N has $\dim N = n$ and $\omega(X, Y) = 0$, for every X, Y tangent to N. Equivalently, a Lagrangian foliation is a foliation whose tangent distribution is a Lagrangian distribution. A Lagrangian foliation is also called a *polarization* [7] and an *integrable almost cotangent structure* (see [6, p 322]).

(c) (M, ω) is said to be endowed with an *almost bi-Lagrangian structure* if M has two transversal Lagrangian distributions $\mathcal{D}_1$ and $\mathcal{D}_2$.

(d) $(M, \omega, \mathcal{F}_1, \mathcal{F}_2)$ is said to be a *bi-Lagrangian manifold* if the tangent distributions $\mathcal{D}_i = T(\mathcal{F}_i)$, $i = 1, 2$, define an almost bi-Lagrangian structure.

As is well known, a symplectic manifold (M, ω) admits several symplectic connections (a *symplectic connection* ∇ is a torsionless connection parallelizing ω), but one needs additional assumptions to obtain a *canonical connection* (see [8]), where some sufficient conditions are quoted). Bi-Lagrangian manifolds admit a canonical connection, introduced by Hess in [1] in a quite difficult way, that one can reduce to the following expression (see also [4, 5]):

(e) The *canonical connection of a bi-Lagrangian manifold* is the unique symplectic connection ∇ which parallelizes both foliations $\mathcal{F}_1$ and $\mathcal{F}_2$, i.e. such that $\nabla_X Y \in T(\mathcal{F}_i)$, for all vector fields X in M and all vector fields $Y \in T(\mathcal{F}_i)$.

On the other hand, we shall need some notation for almost product structures:

(f) An *almost product manifold* (M, F) is a manifold M endowed with a (1, 1) tensor field F satisfying $F^2 = \mathrm{id}$. Then $F^+ = \{X \in \mathcal{X}(M)/F(X) = X\}$ and $F^- = \{X \in \mathcal{X}(M)/F(X) = -X\}$ define two transversal distributions, i.e. the tangent bundle of M admits a decomposition as a Whitney sum: $TM = F^+ \oplus F^-$. Moreover, the converse is true: if a manifold M is endowed with two transversal distributions $TM = \mathcal{D}_1 \oplus \mathcal{D}_2$, then one can define the projectors $\pi_i : TM \to \mathcal{D}_i$, and then $F = \pi_1 - \pi_2$ defines an almost product structure on M.

The involutiveness of both distributions F^+ and F^- can be easily characterized by the vanishing of the Nijenhuis tensor field N_F of F (remember that the *Nijenhuis tensor field*

N_F is the (1, 2) tensor field defined as $N_F(X, Y) = [FX, FY] + F^2[X, Y] - F[FX, Y] - F[X, FY]$).
On the other hand, one can easily check that a linear connection ∇ on an almost product manifold (M, F) parallelizes both distributions F^+ and F^- iff $\nabla F = 0$. Moreover, one can prove [9] that if ∇ is a torsionless connection parallelizing F, then $N_F = 0$, thus proving that the manifold has two transversal foliations.

(g) A *Riemannian almost product manifold* (M, F, G) is an almost product manifold (M, F) endowed with a Riemannian metric G such that $G(FX, FY) = G(X, Y)$, for all $X, Y \in \mathcal{X}(M)$. In this case the distributions F^+ and F^- are G-orthogonal.

We assume that the basic theory of complex and Kähler manifolds is known.

3. Lagrangian distributions

Let (M, ω) be a symplectic manifold, with $\dim M = 2n$. First, we shall prove the following result.

Theorem 1. *Let (M, ω) be a symplectic manifold and let $\mathcal{D}$ be a Lagrangian distribution. Then, M admits infinitely many different Lagrangian distributions.*

Proof. First of all, we shall prove that there exists a transversal Lagrangian distribution. Taking into account that (M, ω) is an almost symplectic manifold one can find [7, 10] an almost Hermitian structure (J, G) on M such that $\omega(X, Y) = G(JX, Y)$. Let $\mathcal{D}^\perp$ be the G-orthogonal distribution to $\mathcal{D}$. Then one has:

(a) If $X, Y \in \mathcal{D}$, then $G(JX, Y) = \omega(X, Y) = 0$, thus proving that $J(\mathcal{D}) = \mathcal{D}^\perp$;
(b) $\mathcal{D}^\perp$ is a Lagrangian distribution, because $\omega(JX, JY) = \omega(X, Y)$, for all $X, Y \in \mathcal{X}(M)$.

Let F be the almost product structure defined by $\mathcal{D}$ and $\mathcal{D}^\perp$, i.e. $F^+ = \mathcal{D}$ and $F^- = \mathcal{D}^\perp$. Then, one can easily check that $J \circ F = -F \circ J$. Moreover, one can prove that (M, F, G) is a Riemannian almost product manifold:

If $X \in \mathcal{X}(M)$, then $X = X_1 + X_2$, where $X_1 \in F^+ = \mathcal{D}$ and $X_2 \in F^- = \mathcal{D}^\perp = J(\mathcal{D})$, and one can write $X_2 = J(X_3)$, with $X_3 \in F^+$. Using this notation we obtain

$$
\begin{aligned}
G(X, Y) &= G(X_1 + JX_3, Y_1 + JY_3) = G(X_1, Y_1) + G(JX_3, JY_3) \\
&= G(X_1, Y_1) + G(X_3, Y_3)
\end{aligned}
$$

and

$$
\begin{aligned}
G(FX, FY) &= G(X_1 - JX_3, Y_1 - JY_3) = G(X_1, Y_1) + G(JX_3, JY_3) \\
&= G(X_1, Y_1) + G(X_3, Y_3)
\end{aligned}
$$

thus proving $G(FX, FY) = G(X, Y)$.

Let α and β be real numbers such that $\alpha^2 + \beta^2 = 1$. Then one can define the (1, 1) tensor field $F_{(\alpha,\beta)}$ given by $F_{(\alpha,\beta)}(X) = \alpha F(X) + \beta JF(X)$. Then, one easily check that $F_{(\alpha,\beta)} \circ F_{(\alpha,\beta)} = \mathrm{id}$, thus proving that $F_{(\alpha,\beta)}$ defines an almost product structure on M.

Let us consider the distribution $F^+_{(\alpha,\beta)}$. We shall show that it is a Lagrangian distribution. Let us consider $X, Y \in F^+_{(\alpha,\beta)}$. Then, $F_{(\alpha,\beta)}(X) = X$ and $F_{(\alpha,\beta)}(Y) = Y$, i.e. $X = \alpha F(X) + \beta JF(X)$ and $Y = \alpha F(Y) + \beta JF(Y)$. Let us compute $\omega(X, Y)$:

$$
\begin{aligned}
\omega(X, Y) &= G(JX, Y) = G(\alpha JFX + \beta JJFX, \alpha FY + \beta JFY) \\
&= \alpha^2 G(JFX, FY) + \alpha\beta G(JFX, JFY) + \alpha\beta G(-FX, FY) + \beta^2 G(-FX, JFY)
\end{aligned}
$$

$$= -\alpha^2 G(JX, Y) + \alpha\beta G(X, Y) - \alpha\beta G(X, Y) + \beta^2 G(X, JY)$$

$$= -\alpha^2 \omega(X, Y) + \beta^2 G(JY, X) = -\alpha^2 \omega(X, Y) - \beta^2 \omega(X, Y) = -\omega(X, Y)$$

thus proving $\omega(X, Y) = 0$ and finishing the proof. □

Remark. The above construction shows that M admits a *biparacomplex structure* in the sense of an unpublished paper of Cruceanu, which is explained in a paper by Santamaría [9] (there are other equivalent presentations [11]), because one has two almost product structures F and $P = J \circ F$ which anticommutates ($F \circ P = -P \circ F$). A *biparacomplex manifold* M is, equivalently, a manifold endowed with three equidimensional distributions, $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3$, such that the tangent bundle of M admits three decompositions as a Whitney sum $TM = \mathcal{D}_1 \oplus \mathcal{D}_2 = \mathcal{D}_1 \oplus \mathcal{D}_3 = \mathcal{D}_2 \oplus \mathcal{D}_3$. When these distributions are involutive the manifold is said to be endowed with a *3-web*.

On the other hand, the deep relations among almost Hermitian, almost para-Hermitian and biparacomplex structures on a manifold have been studied by the authors [12].

If (M, J, G) is a Kähler manifold, then the Kähler form ω, defined as $\omega(X, Y) = G(JX, Y)$, is closed and then (M, ω) is a symplectic manifold. One can expect that Lagrangian foliations in a Kähler manifold possess strong properties. This is the case, because one has:

Theorem 2. *Let $\mathcal{F}$ be a Lagrangian foliation in a Kähler manifold (M, J, G), such that the Levi-Civita connection ∇ of G parallelizes the foliation. Then:*

(a) the orthogonal distribution $\mathcal{D}^\perp = (T\mathcal{F})^\perp$ is parallel respect to ∇;
(b) $\mathcal{D}^\perp$ is involutive, and then M is a bi-Lagrangian manifold;
(c) ∇ is the canonical connection associated with the bi-Lagrangian structure;
(d) all the distributions obtained in the above theorem are involutive.

Proof.

(a) Let $X \in \mathcal{D} = T(\mathcal{F})$, $Y \in \mathcal{X}(M)$ and $Z \in \mathcal{D}^\perp$ and let ∇ be the Levi-Civita connection of G. Then, one obtains

$$0 = (\nabla_Y G)(X, Z) = Y(G(X, Z)) - G(\nabla_Y X, Z) - G(X, \nabla_Y Z) = 0 - 0 - G(X, \nabla_Y Z)$$

thus proving that $\nabla_Y Z \in \mathcal{D}^\perp$.

(b) Let $A, B \in \mathcal{D}^\perp$. We must prove that $[A, B] \in \mathcal{D}^\perp$. Let $X, Y \in \mathcal{D}$ such that $J(X) = A$, $J(Y) = B$ (the existence and uniqueness of X, Y can be deduced from the isomorphism $J|_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}^\perp$). Then as (M, J, G) is Kähler, one has that ∇ is torsionless and $\nabla J = 0$, thus allowing us to obtain

$$[A, B] = [JX, JY] = \nabla_{JX} JY - \nabla_{JY} JX = J(\nabla_{JX} Y - \nabla_{JY} X) \in J(\mathcal{D}) = \mathcal{D}^\perp.$$

(c) (M, J, G) being Kähler, one has $\nabla\omega = 0$ and ∇ is torsionless. By the hypothesis, ∇ parallelizes $\mathcal{F}$, and by (1) and (2) ∇ also parallelizes $\mathcal{F}^\perp$, where $\mathcal{F}^\perp$ denotes the foliation defined by $\mathcal{D}^\perp$.

(d) We have to prove that $X, Y \in F^+_{(\alpha,\beta)}$ implies $[X, Y] \in F^+_{(\alpha,\beta)}$. The direct computation is tedious, so we shall give a different proof. Taking into account (M, J, G) is Kähler and (3) one has $\nabla J = 0$ and $\nabla F = 0$. Let $P = J \circ F$. Then, one easily proves that $\nabla P = 0$ (remember the remark below the above proposition: P is an almost product structure). One can also write $F_{(\alpha,\beta)}(X) = \alpha F(X) + \beta P(X)$ and one has

$$(\nabla F_{(\alpha,\beta)})(Y, X) = \nabla_X (F_{(\alpha,\beta)} Y) - F_{(\alpha,\beta)}(\nabla_X Y) = \nabla_X(\alpha F Y + \beta P Y) - F_{(\alpha,\beta)}(\nabla_X Y)$$

$$= \alpha F(\nabla_X Y) + \beta P(\nabla_X Y) - F_{(\alpha,\beta)}(\nabla_X Y) = 0.$$

Then, $\nabla F_{(\alpha,\beta)} = 0$, and then $N_{F_{(\alpha,\beta)}} = 0$ thus proving that the distributions $F^{+}_{(\alpha,\beta)}$ and $F^{-}_{(\alpha,\beta)}$ are both involutive. □

4. Bi-Lagrangian manifolds

A bi-Lagrangian manifold is a symplectic manifold endowed with two transversal Lagrangian foliations. As we have said, such a manifold admits a canonical symplectic connection which preserves both foliations. We want to prove that this connection is the Levi-Civita connection of a canonical metric on the manifold.

Theorem 3. *Let $(M, \omega, \mathcal{F}_1, \mathcal{F}_2)$ be a bi-Lagrangian manifold. Then M admits a canonical neutral metric g whose Levi-Civita connection coincides with the canonical connection of the bi-Lagrangian manifold.*

Proof.

(a) Let $(M, \omega, \mathcal{F}_1, \mathcal{F}_2)$ be a bi-Lagrangian manifold. We define the (1, 1) tensor field F by $F|_{\mathcal{D}_1} = \mathrm{id}$ and $F|_{\mathcal{D}_2} = -\mathrm{id}$, let $\mathcal{D}_i$ be the tangent distribution to the foliation $\mathcal{F}_i$, and the map g which applies two vector fields $X, Y \in \mathcal{X}(M)$ to $g(X, Y) = \omega(FX, Y)$. Now, we prove that g is a neutral metric.
Let $x \in M$ and let $\{X_1, \ldots, X_n, X_{n+1}, \ldots, X_{2n}\}$ be a basis of the tangent space $T_x M$ such that $\{X_1, \ldots, X_n\}$ generates the first distribution $\mathcal{D}_1$ at x, $\{X_{n+1}, \ldots, X_{2n}\}$ generates the other distribution $\mathcal{D}_2$ at x and the matrix of ω_x is $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ with respect to that basis. One can easily check that $g = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ with respect to the basis, thus proving that g is a neutral metric.

(b) As ∇ is the canonical connection of $(M, \omega, \mathcal{F}_1, \mathcal{F}_2)$ we know that ∇ is a torsionless connection parallelizing ω and both foliations (or equivalently, $\nabla F = 0$). We must prove that $\nabla g = 0$, but this is a direct consequence of the quoted properties of ∇:

$$\begin{aligned}(\nabla_X g)(Y, Z) &= X(g(Y, Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X, Z) \\ &= X(\omega(FY, Z)) - \omega(F(\nabla_X Y), Z) - \omega(FY, \nabla_X Z) \\ &= X(\omega(FY, Z)) - \omega(\nabla_X(FY), Z) - \omega(FY, \nabla_X Z) \\ &= (\nabla_X \omega)(FY, Z) = 0\end{aligned}$$

thus proving the result. □

Let $(M, \omega, \mathcal{F}_1, \mathcal{F}_2)$ be a bi-Lagrangian manifold. A diffeomorphism $\varphi : M \to M$, which preserves the symplectic structure (i.e. such that $\varphi^*\omega = \omega$) and the foliations (i.e. φ_* sends a tangent vector to $\mathcal{F}_i$ to tangent vectors to the same foliation $\mathcal{F}_i$) can be called a *bi-Lagrangian automorphism*. One can prove:

Corollary 4. *Let $(M, \omega, \mathcal{F}_1, \mathcal{F}_2)$ be a bi-Lagrangian manifold. If $\varphi : M \to M$ is a bi-Lagrangian automorphism, then φ is an isometry respect to the canonical neutral metric of M.*

Proof. φ being a bi-Lagrangian automorphism one has $F \circ \varphi_* = \varphi_* \circ F$ and then

$$\begin{aligned}g(\varphi_* X, \varphi_* Y) &= \omega(F\varphi_* X, \varphi_* Y) = \omega(\varphi_* FX, \varphi_* Y) \\ &= (\varphi^*\omega)(FX, Y) = \omega(FX, Y) = g(X, Y).\end{aligned}$$

□

Appendix. Para-Kähler geometry

Theorem 3 can be extended to the following: *there exists a bijection between bi-Lagrangian structures and para-Kähler structures on a manifold*. In order to understand this result, we shall show some properties of para-complex and para-Kähler geometry, which have been developed over the last five decades, starting with the works of Rashevskij [13] and Libermann [14] (see the survey of Cruceanu *et al* [15] and the more than 100 references therein). The general framework of this theory is the geometry of almost product structures (see definition (f) of section 2). The basic notions that we need are the following.

(a) An *almost para-Hermitian manifold* (M, F, g) is a manifold M endowed with an almost product structure F and a semi-Riemannian metric g such that $g(FX, FY) = -g(X, Y)$, for all vector fields X, Y in M. Then F^+ and F^- define two equidimensional distributions isotropic respect to g, and g is a neutral metric, i.e. the signature of g is (n, n). One can define an almost symplectic structure on M given by the 2-form ω defined by $\omega(X, Y) = g(FX, Y)$.
(b) An *almost para-Kähler manifold* is an almost para-Hermitian manifold whose almost symplectic form ω is closed, i.e. (M, ω) is a symplectic manifold.
(c) A *para-Kähler manifold* is an almost para-Hermitian manifold (M, F, g) such that $\nabla F = 0$, where ∇ is the Levi-Civita connection of g. Equivalently, both distributions F^+ and F^- are involutive and ω is closed.

As one can see both theories of complex and para-complex manifolds are quite similar. Nevertheless, there exist important differences between them. In particular, the metric of an almost para-Hermitian manifold is neutral and not Riemannian. On the other hand, the geometric motivation of para-complex geometry is the study of manifolds endowed with two transversal distributions, thus it is natural to think that this geometry would be useful in the study of bi-Lagrangian manifolds.

Then, one can easily prove the following:

Theorem 5. *Let M be a manifold. There exists a bijection between almost bi-Lagrangian structures on M and almost para-Kähler structures on M.*

Proof.

(a) In theorem 3 we have proved that an almost bi-Lagrangian manifold $(M, \omega, \mathcal{D}_1, \mathcal{D}_2)$ admits a neutral metric g satisfying $g(X, Y) = \omega(FX, Y)$, for all vector fields $X, Y \in \mathcal{X}(M)$. Then, one can consider the almost product structure F defined by the given distributions $\mathcal{D}_1$ and $\mathcal{D}_2$, proving that (M, F, g) is an almost para-Hermitian manifold:

$$g(FX, FY) = \omega(F^2X, FY) = \omega(X, FY) = -\omega(FY, X) = -g(Y, X) = -g(X, Y).$$

Finally, the almost symplectic form attached to the almost para-Hermitian manifold (M, F, g) is ω, which is closed, thus proving that (M, F, g) is almost para-Kähler.
(b) Let (M, F, g) be an almost para-Kähler manifold. Then, one can easily prove that F^+ and F^- are Lagrangian distributions with respect to ω (where $\omega(X, Y) = g(FX, Y)$), because F^+ and F^- are g-isotropic. □

The above bijection is restricted to bi-Lagrangian and para-Kähler structures.

Theorem 6. *Let M be a manifold. There exists a bijection between bi-Lagrangian structures on M and para-Kähler structures on M.*

Proof.

(a) Let $(M, \omega, \mathcal{F}_1, \mathcal{F}_2)$ be a bi-Lagrangian manifold and let (M, F, g) be the almost para-Kähler structure defined by the above theorem. Then (M, F, g) is a para-Kähler manifold, because both distributions $F^+ = T(\mathcal{F}_1)$ and $F^- = T(\mathcal{F}_2)$ are, obviously, involutive.

(b) Let (M, F, g) be a para-Kähler manifold and let (M, ω, F^+, F^-) be the almost bi-Lagrangian manifold obtained by the above theorem. Then, ω is closed and F^+, F^- are involutive thus proving that (M, ω, F^+, F^-) is a bi-Lagrangian manifold. □

Final comments

The equivalence between bi-Lagrangian and para-Kähler structures seems to be known (see p 97 of the survey [15] or a paper of Kaneyuki [16]), but it has not been shown in an explicit way. Moreover, the canonical metric of a bi-Lagrangian manifold has not been used in symplectic geometry. For this reason, we have shown this equivalence in a self-contained way. On the other hand, the canonical connection of a bi-Lagrangian structure, was introduced by Hess in a quite obscure way. Vaisman [4] and Boyom [5] subsequently obtained a nice expression for this connection (see definition (e) of section 2). In the present paper, we have proved that it is a Levi-Civita connection, which clarifies its nature. Moreover, all the results concerning para-Kähler geometry can be translated to bi-Lagrangian geometry.

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